MA 26500 Midterm Exam 1 Spring 2021

Name: Solutions

ID number: ______________________

Instructions:

1. You have 50 minutes to complete this exam.
2. There are 10 problems on this exam. Eight are multiple choice and two of them are free-response problems.
3. **Circle** one **and only one** option for each multiple-choice problem. No partial credit will be given for multiple-choice problems.
4. Show **all** relevant work on free-response problems. Partial credit will be given for clear steps leading to solutions. **Little to no credit will be given for little to no work.**
5. No books, notes, or calculators are allowed.
6. Please turn off your cell phone.

Purdue University faculty and students commit themselves towards maintaining a culture of academic integrity and honesty. The students taking this exam are not allowed to seek or obtain any kind of help from anyone to answer questions on this test. If you have questions, consult only an instructor or a proctor. You are not allowed to look at the exam of another student. You may not compare answers with anyone else or consult another student until after you finish your exam and hand it in to a proctor or to an instructor. You may not consult notes, books, calculators, cameras, or any kind of communications devices until after you finish your exam and hand it in to a proctor or to an instructor. If you violate these instructions you will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe and may include an F in the course. All cases of academic dishonesty will be reported to the Office of the Dean of Students. Your instructor and proctors will do everything they can to stop and prevent academic dishonesty during this exam. If you see someone breaking these rules during the exam, please report it to the proctor or to your instructor immediately. Reports after the fact are not very helpful.

I agree to abide by the instructions above:

Signature: ______________________

1. Circle either True (T) or False (F) for the following statements:

$$\begin{cases} x_1 + x_2 = 1 \\ 2x_1 + 2x_2 = 2 \\ 3x_1 + 3x_2 = 3 \end{cases}$$ has a solution

T / **F** : If a system of linear equations has more equations than variables, it must be inconsistent.

T / **F** : If a system of linear equations has more variables than equations, it must have infinitely many solutions.

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + x_2 + x_3 = 2 \end{cases}$$ has no solution

T / **F** : If A is an invertible square matrix, then $\det A^{-1} = \frac{1}{\det A}$.

$\det(A^{-1})\det(A) = \det(I) = 1$

T / **F** : If A and B are square matrices, $\det(A+B) = \det A + \det B$.

T / **F** : Every matrix transformation is a linear transformation.

2. Let

$$A = \begin{bmatrix} 1 & 0 & 5 & 0 & -1 \\ 0 & 1 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 pivots

2 free variables

What are the rank of A and the dimension of the Null Space of A?

A. $\text{rank}(A) = 3$ and $\dim \text{Nul}(A) = 2$

B. $\text{rank}(A) = 1$ and $\dim \text{Nul}(A) = 1$

C. $\text{rank}(A) = 2$ and $\dim \text{Nul}(A) = 3$

D. $\text{rank}(A) = 3$ and $\dim \text{Nul}(A) = 3$

E. $\text{rank}(A) = 4$ and $\dim \text{Nul}(A) = 1$

F. $\text{rank}(A) = 2$ and $\dim \text{Nul}(A) = 2$

3. Which one of the following sets is a basis for $\mathbb{R}^4$?

dim $\mathbb{R}^4 = 4$ so must have 4 vectors!

Not LI

$v_1 + v_2 = v_4$

~~A.~~ $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 8 \\ 7 \end{bmatrix} \right\}$

3 vectors

~~B.~~ $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 2 \\ 4 \\ -2 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \right\}$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ -2 & 0 & 0 & 2 \\ 8 & 2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 8 & 2 & 5 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix}$$

det $\neq 0$

so linearly independent

Not LI

$v_1 + v_2 + v_3 = v_4$

~~D.~~ $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 2 \\ 7 \end{bmatrix} \right\}$

3 vectors

~~E.~~ $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \\ 8 \\ 2 \end{bmatrix} \right\}$

5 vectors

~~F.~~ $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ -3 \\ -2 \end{bmatrix} \right\}$

4. Which of the following is **not** a subspace of $\mathbb{R}^3$?

A. $\text{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\end{bmatrix}\right\}$ ✓

B. $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$ ✓

C. The plane $2x + 2y - z = 0$ ✓

D. $\text{Col}\left(\begin{bmatrix}5 & 4 & 23 & 0\\17 & 1 & 0 & 3\\2 & 3 & -1 & 1\end{bmatrix}\right)$ ✓

E. The plane $z = 2$ ✗

F. $\text{Nul}\left(\begin{bmatrix}6 & 18 & 9\\1 & 2 & 4\end{bmatrix}\right)$ ✓

planes that pass through the origin are subspaces.

$z=2$ doesn't contain the zero vector!

5. Find the third column of C given that

$$B = \begin{bmatrix}1 & 2\\-2 & 3\end{bmatrix} \text{ and } BC = \begin{bmatrix}7 & 7 & 5 & -2\\7 & 14 & 4 & -3\end{bmatrix}$$

$$Bx = \begin{bmatrix}5\\4\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & 2 & 5\\-2 & 3 & 4\end{array}\right] \to \left[\begin{array}{cc|c}1 & 2 & 5\\0 & 7 & 14\end{array}\right] \to \left[\begin{array}{cc|c}1 & 2 & 5\\0 & 1 & 2\end{array}\right]$$

$$\to \left[\begin{array}{cc|c}1 & 0 & 1\\0 & 1 & 2\end{array}\right]$$

A. $\begin{bmatrix}2\\1\end{bmatrix}$

B. $\begin{bmatrix}5\\4\end{bmatrix}$

C. $\begin{bmatrix}-4\\5\end{bmatrix}$

D. $\begin{bmatrix}1\\2\end{bmatrix}$

E. $\begin{bmatrix}-1\\-2\end{bmatrix}$

F. $\begin{bmatrix}23/7\\-6/7\end{bmatrix}$

6. Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2a-g & 2b-h & 2c-i \\ \frac{1}{3}g & \frac{1}{3}h & \frac{1}{3}i \\ d & e & f \end{bmatrix}$$

Given that $\det(A) = 6$, what is $\det(B)$?

A. 6

B. -6

C. 4

D. -4

E. 9

F. -9

$$A \longrightarrow \begin{bmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & i \end{bmatrix} \longrightarrow \begin{bmatrix} 2a-g & 2b-h & 2c-i \\ d & e & f \\ g & h & i \end{bmatrix}$$

$\det A$ $\qquad 2\det A \qquad 2\det A$

$$\longrightarrow \begin{bmatrix} 2a-g & 2b-h & 2c-i \\ g & h & i \\ d & e & f \end{bmatrix} \longrightarrow \begin{bmatrix} 2a-g & 2b-h & 2c-i \\ \frac{1}{3}g & \frac{1}{3}h & \frac{1}{3}i \\ d & e & f \end{bmatrix}$$

$-2\det A \qquad -\frac{2}{3}\det A$

$$\det B = -\frac{2}{3}\det A = -\frac{2}{3}(6) = -4$$

7. Consider the basis $\beta = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\}$ of $\mathbb{R}^2$. What is the coordinate vector of $\begin{bmatrix} -7 \\ 5 \end{bmatrix}$ with respect to β?

A. $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

B. $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$

C. $\begin{bmatrix} -7 \\ 5 \end{bmatrix}$

D. $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$

E. $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$

F. $\begin{bmatrix} -5 \\ -4 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & -3 & -7 \\ -3 & 5 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -3 & -7 \\ 0 & -4 & -16 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -3 & -7 \\ 0 & 1 & 4 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 4 \end{array}\right]$$

8. Compute the $(3, 2)$ entry of the inverse of

$$A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 0 & 0 \\ 6 & 8 & -2 \end{bmatrix}$$

A. 1/19

B. -1/19

C. 1/13

D. -1/13

E. -2/19

F. 2/19

$$C_{23} = (-1)^{2+3} \det \begin{bmatrix} 2 & 3 \\ 6 & 8 \end{bmatrix} = (-1)(16 - 18) = 2$$

$$\det A = (-1) \cdot - \det \begin{bmatrix} 3 & 4 \\ 8 & -2 \end{bmatrix} = (-6 - 32) = -38$$

(expanding along Row 2)

$$(A^{-1})_{32} = \frac{C_{23}}{\det A} = \frac{2}{-38} = \boxed{-\frac{1}{19}}$$

9. Are the following vectors linearly independent or linearly dependent? If they are linearly dependent, provide an equation of linear dependence.

$$\left\{ \begin{bmatrix} 4 \\ -8 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 4 \end{bmatrix} \right\}$$

$$\left[\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{\hphantom{4 \quad 2 \quad 4}}^{A}} & \\ 4 & 2 & 4 & 0 \\ -8 & 6 & -3 & 0 \\ 2 & 5 & 4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1/2 & 1 & 0 \\ -8 & 6 & -3 & 0 \\ 2 & 5 & 4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1/2 & 1 & 0 \\ 0 & 10 & 5 & 0 \\ 0 & 4 & 2 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1/2 & 1 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 4 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3/4 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$Ax=0$ has infinitely many solutions (not just trivial soln $x=0$) so this set is linearly dependent. Moreover since the solution set to $Ax=0$ is

$$\begin{cases} x_1 = -3/4\, x_3 \\ x_2 = -1/2\, x_3 \\ x_3 = \text{free} \end{cases} \qquad \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\} = \left\{ x_3 \begin{bmatrix} -3/4 \\ -1/2 \\ 1 \end{bmatrix} \right\}$$

an equation of linear dependence is

$$-\frac{3}{4} \begin{bmatrix} 4 \\ -8 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \\ 4 \end{bmatrix} = 0$$

10. Find a basis for the null space of A, Nul A if

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -2 & 1 & 4 \\ 0 & 1 & -2 & 1 & 2 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -2 & 1 & 4 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 & 2 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

x_3, x_4 free

$$\begin{cases} x_1 + 7x_3 + 2x_4 = 0 \\ x_2 - 2x_3 + x_4 = 0 \\ x_5 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -7x_3 - 2x_4 \\ x_2 = 2x_3 - x_4 \\ x_3 = x_3 \text{ (free)} \\ x_4 = x_4 \text{ (free)} \\ x_5 = 0 \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -7 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} -7 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$